Method for computing long periodic orbits of dynamical systems

Lambros Drossos, Omiros Ragos, Michael N. Vrahatis, and Tassos Bountis

Department of Mathematics, University of Patras, GR-261.10 Patras, Greece

(Received 19 June 1995)

The accurate computation of periodic orbits, particularly those of long period, is very important for studying a number of interesting properties of dynamical systems. In this paper, we implement a method for computing periodic orbits of dynamical systems efficiently and to a high degree of accuracy. This method converges rapidly, within relatively large regions of initial conditions, and is independent of the local dynamics near periodic points. The only computable information required is the signs of various function evaluations carried out during the integration of the equations of motion. Here we apply this method to a Duffing oscillator and illustrate its advantages by comparing it with other widely used perturbation techniques.

I. INTRODUCTION

In the study of dynamical systems the computation of periodic solutions and the knowledge of their stability properties are of great importance. In particular, there are fundamental problems of nonlinear dynamics, such as the period-doubling route to chaos [1] and the breakup of quasiperiodic motion [2], which make crucial use of this information for periodic orbits of arbitrarily long period.

As is well known, the more classical and widely used approaches to this problem are perturbative in character and essentially rely on a systematic search of phase space in order to locate the initial conditions for each orbit. In many dynamical systems, however, such a search becomes unsuccessful or too time consuming to be of practical use, especially in the case of orbits of very long period.

In this paper, we apply an efficient method whose convergence properties are independent of the local dynamics near the periodic orbit. This method is especially useful in the case of *conservative* dynamical systems, which possess many periodic orbits, often of the same period, close to each other in phase space.

In Sec. II we briefly review the essential features of other, widely used perturbation methods for computing periodic orbits. The above-mentioned methods rely on the computation of the monodromy matrix of the *linearized* equations around any point and compute the appropriate corrections, using information taken from this matrix. This introduces additional inaccuracies caused by the local properties of the dynamics near the points of periodic orbits. In the same section we also describe the determination of the stability of periodic orbits by using the eigenvalues of the Jacobian of the Poincaré return map at any point of these orbits.

In Sec. III, we briefly present an efficient method for computing periodic orbits. According to this method, we use a characteristic polyhedron, surrounding a point of the orbit, and by successive refinements of this polyhedron we calculate the desired periodic orbit. The only information needed for this refinement is the signs of vari-

ous function evaluations on the vertices of the polyhedron. More details of this method can be found in [3,4].

In Sec. IV, we apply and compare the above methods to the computation of long periodic orbits of a driven conservative Duffing oscillator. We demonstrate that our method is more efficient, by showing that it has significantly larger regions of convergence than the perturbation methods.

II. PERTURBATION METHODS FOR COMPUTING PERIODIC ORBITS

Let us consider a conservative dynamical system of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) , \tag{1}$$

with $\mathbf{x} = (x, \dot{x}) \in \mathbb{R}^2$ and $\mathbf{f} = (f_1, f_2)$ periodic in t with frequency ω. We shall look for initial conditions of periodic orbits of period p of system (1) that intersect the surface of section

$$\Sigma_{t_0} = \{(x(t_k) , \dot{x}(t_k)) \text{ with } t_k = t_0 + k\frac{2\pi}{\omega} , \quad k \in \mathbb{N}\} , \tag{2}$$

at a finite number of points p. Thus the dynamics is studied in connection with a Poincaré map $\Phi^p = P_{t_0} : \Sigma_{t_0} \rightarrow \Sigma_{t_0}$, constructed by following the solutions of (1) in continuous time. According to the usual perturbation methods [5,6], one starts with an initial vector $\mathbf{x}_0^{(0)} \in \Sigma_{t_0}$ near a desired periodic orbit and then iteratively estimates successive corrections $\delta\mathbf{x}_0^{(0)}$, $\delta\mathbf{x}_0^{(1)}$, $\delta\mathbf{x}_0^{(2)}$, ..., such that the sequence of points

$$\mathbf{x}_0^{(i+1)} = \mathbf{x}_0^{(i)} + \delta\mathbf{x}_0^{(i)} , \quad \mathbf{x}_0^{(i)} \in \Sigma_{t_0} , \quad i = 0, 1, \ldots \tag{3}$$

will eventually converge to $\lim_{i \rightarrow \infty} \mathbf{x}_0^{(i)} = \mathbf{x}_0^*$ with

$$\mathbf{x}(\mathbf{x}_0^*, 0) = \mathbf{x}\left[\mathbf{x}_0^* , \quad T = p\frac{2\pi}{\omega}\right] . \tag{4}$$

The latter is the periodicity condition. T is the total time

needed for the orbit to return to the point $\mathbf{x}_0^*$.

Supposing that the $(i+1)$th approximation gives the periodic solution and applying Taylor's formula to both sides of (4) we get that, up to first-order terms, the corrections required to obtain this approximation from the previous one are the solutions of the system

$$x(\mathbf{x}_0^{(i)},0)+\delta x_0^{(i)}=x(\mathbf{x}_0^{(i)},T)+\frac{\partial x}{\partial x_0}\delta x_0^{(i)}+\frac{\partial x}{\partial \dot{x}_0}\delta \dot{x}_0^{(i)} \ ,$$

$$\dot{x}(\mathbf{x}_0^{(i)},0)+\delta \dot{x}_0^{(i)}=\dot{x}(\mathbf{x}_0^{(i)},T)+\frac{\partial \dot{x}}{\partial x_0}\delta x_0^{(i)}+\frac{\partial \dot{x}}{\partial \dot{x}_0}\delta \dot{x}_0^{(i)} \ ,$$

$$(5)$$

where the partial derivatives correspond to the ith approximation of the periodic point $\mathbf{x}_0^{(i)}$ and are evaluated at $t=T$. These partial derivatives can be computed by integrating the variational equations of system (1):

$$\frac{d}{dt}\left[\frac{\partial \mathbf{x}}{\partial x_0}\right]=\frac{\partial \mathbf{f}}{\partial x}\frac{\partial x}{\partial x_0}+\frac{\partial \mathbf{f}}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial x_0} \ ,$$

$$\frac{d}{dt}\left[\frac{\partial \mathbf{x}}{\partial \dot{x}_0}\right]=\frac{\partial \mathbf{f}}{\partial x}\frac{\partial x}{\partial \dot{x}_0}+\frac{\partial \mathbf{f}}{\partial \dot{x}}\frac{\partial \dot{x}}{\partial \dot{x}_0} \ .$$

$$(6)$$

We name the above-mentioned technique the *variational perturbation method* (VPM). The derivatives can also be approximated by the relations

$$\frac{\partial \mathbf{x}}{\partial x_0}\simeq \frac{\mathbf{x}(x_0+h,\dot{x}_0)-\mathbf{x}(x_0,\dot{x}_0)}{h} \ ,$$

$$\frac{\partial \mathbf{x}}{\partial \dot{x}_0}\simeq \frac{\mathbf{x}(x_0,\dot{x}_0+h)-\mathbf{x}(x_0,\dot{x}_0)}{h} \ ,$$

$$(7)$$

with an appropriately small value of h. We call the corresponding technique the *Euler perturbation method* (EPM).

In both cases the corrections are repeated N times until relation (4) holds within a specified accuracy ϵ, i.e.,

$$|\mathbf{x}_0^{(N)}-\mathbf{x}(\mathbf{x}_0^{(N)} \ , \ T)|\leq \epsilon \ . \qquad (8)$$

Having thus obtained the periodic orbit, we can determine its stability by computing the eigenvalues of the monodromy matrix at this point:

$$\mathscr{A}(\mathbf{x}_0^{(N)},T)=\begin{vmatrix} \dfrac{\partial x}{\partial x_0} & \dfrac{\partial x}{\partial \dot{x}_0} \\[2mm] \dfrac{\partial \dot{x}}{\partial x_0} & \dfrac{\partial \dot{x}}{\partial \dot{x}_0} \end{vmatrix} \ , \qquad (9)$$

where the derivatives of $x,\dot{x}$ are calculated by integrating system (6), or from relations (7), at this approximation.

Now, since (1) is a conservative system, $\det\mathscr{A}(\mathbf{x}_0^{(N)},T)=1$ and (9) has either complex eigenvalues with $|\lambda_i|=1$, $i=1,2$, in which case the periodic orbit is stable (elliptic), or real with, say, $|\lambda_1|>1$, $|\lambda_2|<1$, in which case the periodic orbit is unstable.

III. THE CHARACTERISTIC POLYHEDRON CRITERION AND CHARACTERISTIC BISECTION

In this section, we briefly discuss a method based on the *characteristic polyhedron* concept for the computation

of periodic orbits. A detailed description of this method can be found in [3,4]. First we implement topological degree theory to give a criterion for the existence of a periodic orbit within a given region of the phase space of the system. Then we construct a characteristic polyhedron containing this orbit. Using a generalized bisection method, we iteratively refine this polyhedron to calculate the orbit within a predetermined accuracy.

The problem of finding periodic orbits of period p of dynamical systems in $\mathbb{R}^{n+1}$ amounts to fixing one of the variables, say, $x_{n+1}=$const, and locating points $\mathbf{X}^*=(x_1^*,x_2^*,\ldots,x_n^*)$ on an n-dimensional surface of section Σ_{t_0} that satisfy the equation

$$\Phi^p(\mathbf{X}^*)=\mathbf{X}^* \ , \qquad (10)$$

where $\Phi^p=P_{t_0}:\Sigma_{t_0}\rightarrow\Sigma_{t_0}$ is the Poincaré map of the system. Obviously, this is equivalent to solving the system

$$\mathbf{F}(\mathbf{X})=\mathcal{O} \ , \qquad (11)$$

with $\mathbf{F}=(f_1,f_2,\ldots,f_n)=\Phi^p-I_n$, where I_n is the $n\times n$ identity matrix and $\mathcal{O}=(0,0,\ldots,0)$ is the origin of $\mathbb{R}^n$.

As is known, if we have a function F, which is continuous in a bounded domain $\mathcal{D}$ and the *topological degree of* $\mathbf{F}$ *at* $\mathcal{O}$ *relative to* $\mathcal{D}$ is not equal to zero, then there is at least one solution of system (11) within $\mathcal{D}$ [7]. This criterion can be used, in combination with the construction of a suitable n polyhedron, called the characteristic polyhedron (CP), for the calculation of a solution contained in this region. This can be done as follows. Let $\mathcal{M}_n$ be the $2^n\times n$ matrix whose row are formed by all possible combinations of -1 and 1. Consider now an oriented n polyhedron Π^n, with vertices Υ_k, $k=1,\ldots,2^n$. If the $2^n\times n$ *matrix of signs associated with* $\mathbf{F}$ *and* Π^n, $\mathcal{S}(\mathbf{F};\Pi^n)$, whose entries in the kth row are the vector

$$\mathrm{sgn}\mathbf{F}(\Upsilon_k)=(\mathrm{sgn}f_1(\Upsilon_k),\mathrm{sgn}f_2(\Upsilon_k),\ldots,\mathrm{sgn}f_n(\Upsilon_k)) \ ,$$

$$(12)$$

is identical to $\mathcal{M}_n$, possibly after some permutations of these rows, then Π_n is called the *characteristic polyhedron relative to* $\mathbf{F}$. Furthermore, if $\mathbf{F}$ is continuous, then, under some suitable assumptions on the boundary of Π^n,

$$\deg[\mathbf{F},\Pi^n,\mathcal{O}]=\sum_{\mathbf{X}\in\mathbf{F}^{-1}(\mathcal{O})}\mathrm{sgn}\det\mathscr{A}_{\mathbf{F}}(\mathbf{X})=\pm1\neq0 \qquad (13)$$

(see [8] for a proof), which implies the existence of a periodic orbit inside Π^n. For more details on how to construct a CP and locate a desired periodic orbit see [3,4,9,10].

Next we describe a generalized bisection method that, in combination with the above-mentioned criterion, produces a sequence of characteristic polyhedra of decreasing size always containing the desired solution in order to calculate it within a given accuracy *(characteristic bisection)*. This version of bisection does not require the computation of the topological degree at each step to secure its nonzero value, as others do [11–13]. It can also be applied to problems with imprecise function values since it depends only on their signs. The method simply amounts

to constructing another refined CP, by bisecting a known one, say Π^n. We compute the midpoint **M** of a one-simplex, e.g., $\langle \Upsilon_i, \Upsilon_j \rangle$, which is one edge of Π^n. Then we obtain another CP, Π^n_*, by comparing the sign of **F(M)** with that of $\mathbf{F}(\Upsilon_i)$ and $\mathbf{F}(\Upsilon_j)$ and substituting M for that vertex for which the signs are identical (see [4,8,14,15] for details). Then we continue with another edge. The number of iterations required to obtain a refined characteristic polyhedron Π^n_* whose longest edge length $\Delta(\Pi^n_*)$ satisfies $\Delta(\Pi^n_*) \le \varepsilon$, for some $\varepsilon \in (0,1)$, is given by [8]

$$\zeta = \lceil \log_2[\Delta(\Pi^n)\varepsilon^{-1}] \rceil \tag{14}$$

where the notation $\lceil \ \rceil$ refers to the smallest integer, which is not less than the real number quoted.

IV. NUMERICAL RESULTS

Let us apply these methods to the driven conservative Duffing oscillator described by

$$\ddot{x} = x - x^3 + \alpha \cos\omega t . \tag{15}$$

We integrate numerically Eq. (15), using, e.g., the Bulirsch-Stoer algorithm with appropriate adaptive step-size control [16,17], and compute successive intersection points of the solutions with the surface of section (2), which we plot in Fig. 1 for several initial conditions.

We observe that around the elliptic period-3 periodic orbit, marked by P_1, P_2, P_3 in Fig. 1, there are orbits of higher period, with islands surrounding them, located in a narrow strip near the boundary between the period-3 islands and a large chaotic region of the system. This makes the computation of periodic orbits of high period quite difficult because of the concentration of many periodic orbits in a very small region. In Fig. 2 we plot a magnification of box A of Fig. 1. Here we focus on an orbit of period 51, mark one of its points by P, and enclose it in box B. Magnifying then box B, we observe, in

FIG. 2. Magnification of box A of Fig. 1. Note the period-51 point at P and the islands of period 663 surrounding it, in box B.

Fig. 3, an orbit of period 663 forming a string of 13 islands around the period-51 orbit. Proceeding one step further, we magnify box C of Fig. 3 about one of these 663 periodic points and observe, in Fig. 4, a chain of 9 islands belonging to a period 5967 orbit, etc.

We shall apply and compare the methods described in Secs. II and III to the computation of the above sequence of periodic orbits. The periodic orbits have been calculated with accuracy $\epsilon = 10^{-8}$. The value of h appearing in (7) is taken equal to 10^{-7}.

Let us define the "basin of convergence" of each method by

$$B_i = \{(x_0^{(0)}, \dot{x}_0^{(0)}) \text{ such that } i\text{th method converges to } \mathbf{x}_0^* \},$$
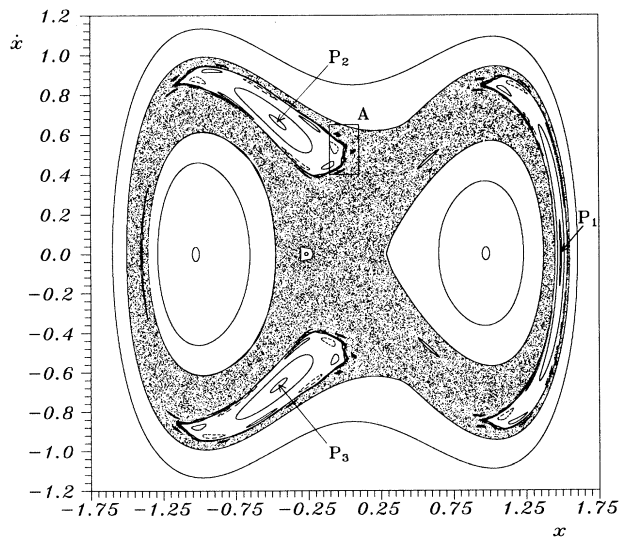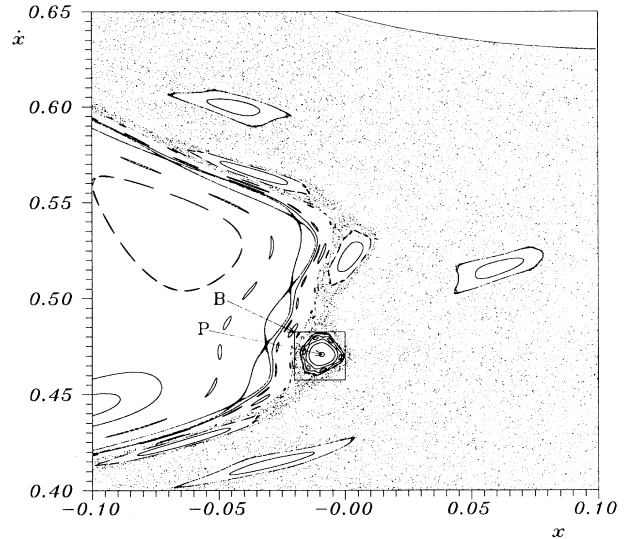
$$\tag{16}$$

FIG. 1. Surface of section points of system (15) for parameters $\alpha = 0.05$, $\omega = 2$, and several initial conditions.
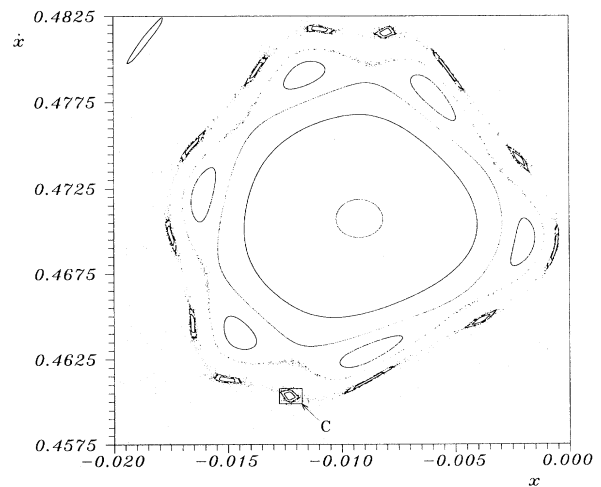

FIG. 3. Magnification of box B of Fig. 2. We observe here a chain of 13 islands of period 663, one of which we enclose in box C.
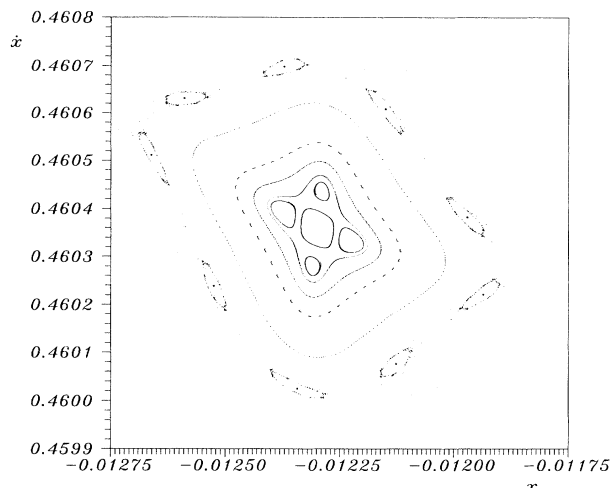
FIG. 4. Magnification of box C of Fig. 3. We show here the period-5967 periodic points at the center of 9 islands around the central point of period 663.

where the numbering $i = 1,2,3$ refers to the VPM, EPM, and CP method (CPM), respectively. Also, we denote by D_i the radius of the largest ball, centered at the periodic orbit such that, for every initial guess within it, the corresponding method converges to this orbit.

What we find is that the CPM always converges to the periodic solution, within any region B_3 that does not contain orbits whose period is a *submultiple* of the period of the desired orbit. This means that the CPM can converge equally well to stable as well as to unstable periodic orbits, independently of where they are located.

The other two methods, the VPM and EPM, behave differently: We have found that the *higher* the period, the *smaller* the regions of convergence B_1 and B_2, in comparison with region B_3 of the CPM. Thus, for periodic or-

bits of large period, it turns out that the VPM and EPM need an initial guess lying within a distance from the solution smaller than the accuracy required for its computation. A comparison between these methods is shown in Table I, where we list the computed points of the periodic orbits x_0^* and the corresponding radii D_1, D_2, and D_3.

In Figs. 5(a)–5(c), we show the basins of convergence (shaded areas) of the VPM and EPM for the orbits of period 27, 51, and 663, discussed above. We observe that the corresponding basins decrease considerably in size as the period of the orbit increases. On the other hand, the convergence regions of the CPM are generally one order of magnitude larger than those of the other two methods and certainly extend over the whole $x, \dot{x}$ region shown in these figures. In fact, the only limitation of the CPM is related to the accuracy of the integration method to the extent that it can ensure the correct calculation of the algebraic signs required.

V. CONCLUSION

A method is proposed for computing long periodic orbits of conservative dynamical systems, using the Poincaré map on a surface of section. This method employs topological degree theory and proceeds by constructing a so-called characteristic polyhedron around a region where a point of the desired periodic orbit is expected to exist. It converges to this point by successively bisecting the sides of the CP. Since this method is independent of the local dynamics of the orbit and relies mainly on the signs of function evaluations at the vertices of the CP, it is more efficient than other more classical methods and enables us to compute orbits of much higher period. Besides, it is globally convergent method, provided that the desired solution is isolated in some region.

We have compared our method with two classical perturbation approaches on the example of a driven Duffing

TABLE I. Comparison of three methods for the computation of periodic orbits x_0^*. The corresponding radii D_1 (VPM), D_2 (EPM) and D_3 (CPM) are also listed (S denotes stable, U unstable).

| Period (stability)	$x_0^* = (x_0^*, \dot{x}_0^*)$	D_1	D_2	D_3
1 (S)	$(-1.024\,572\,46, 0.000\,000\,00)$	0.099	0.099	0.5
2 (S)	$(-0.261\,698\,07, 0.000\,000\,00)$	0.078	0.078	0.1
3 (S)	$(0.453\,504\,79, 0.662\,916\,05)$	0.078	0.078	0.1
12 (S)	$(-0.066\,078\,06, 0.533\,487\,41)$	0.014	0.014	0.048
12 (U)	$(-0.069\,012\,27, 0.485\,169\,47)$	0.016	0.016	0.048
27 (S)	$(0.060\,749\,18, 0.515\,763\,76)$	0.018	0.018	0.061
27 (U)	$(0.028\,231\,33, 0.464\,002\,72)$	0.006	0.006	0.061
51 (S)	$(-0.009\,206\,48, 0.470\,738\,66)$	0.006\,4	0.006\,4	0.023
51 (U)	$(-0.007\,967\,42, 0.494\,051\,15)$	0.002	0.002	0.023
153 (S)	$(-0.014\,535\,18, 0.463\,955\,14)$	0.000\,57	0.000\,57	0.002\,9
153 (U)	$(-0.011\,955\,00, 0.462\,649\,92)$	0.000\,8	0.000\,8	0.002\,9
663 (S)	$(-0.012\,299\,20, 0.460\,359\,10)$	0.000\,28	0.000\,28	0.001\,3
663 (U)	$(-0.013\,465\,63, 0.460\,936\,01)$	0.000\,07	0.000\,07	0.001\,3
5\,967 (S)	$(-0.012\,125\,94, 0.460\,076\,38)$	0.000\,014	0.000\,014	0.000\,33
5\,967 (U)	$(-0.012\,457\,42, 0.460\,125\,23)$	0.000\,001\,4	0.000\,001	0.000\,33
41\,769 (S)	$(-0.012\,163\,80, 0.460\,054\,70)$	0	0	0.000\,006
41\,769 (U)	$(-0.012\,159\,92, 0.460\,049\,80)$	0	0	0.000\,006

DROSSOS, RAGOS, VRAHATIS, AND BOUNTIS
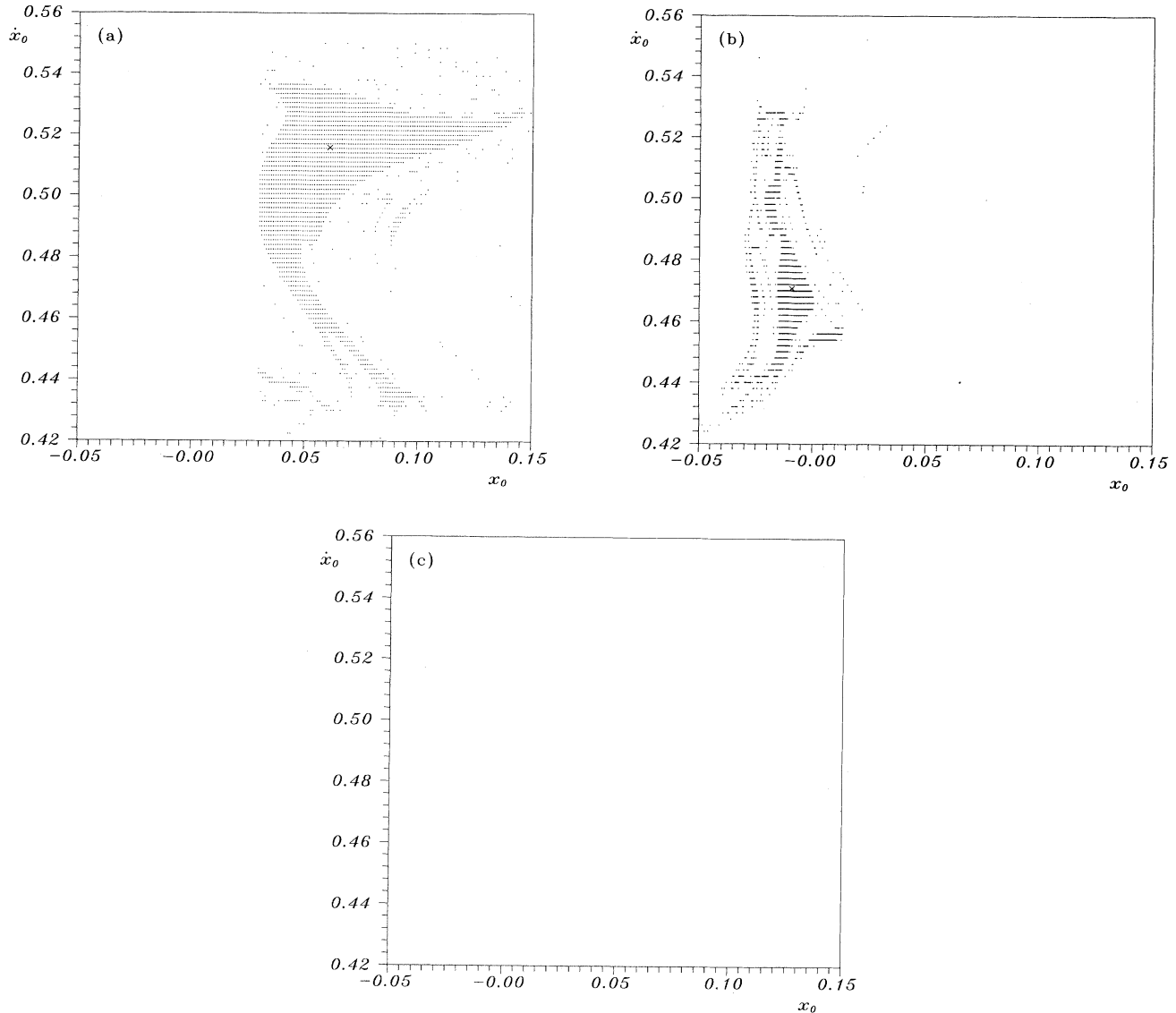
FIG. 5. Basins of convergence (shaded areas) of the VPM, plotted on the same scale, for orbits of (a) period 27, (b) period 51, and (c) period 663. The exact position of the periodic point is marked by ✕. The corresponding regions of the other perturbation method (EPM) do not differ significantly from those of the VPM.

oscillator and have found that the basins of convergence of the perturbation methods are much smaller than those of the CPM and rapidly diminish in size as the period becomes longer. Thus we expect the CPM to be especially useful in cases where a sequence of long periodic orbits is known to exist, such as, e.g., in the problem of approximating invariant surfaces by sequences of periodic orbits in higher-dimensional systems [10], following similar work on area-preserving maps [2]. In this context, interesting results are expected in Hamiltonian systems of three or more degrees of freedom, or $2N$ ($N > 2$) -dimen-

sional symplectic maps to which the CPM has already started to be successfully applied [10].

[1] P. Cvitanovic, *Universality in Chaos,* 2nd ed. (Hilger, Bristol, 1989).

[2] R. MacKay and J. Meiss, *Hamiltonian Dynamical Systems* (Hilger, Bristol, 1986).

[3] M. N. Vrahatis and T. C. Bountis, in *International Conference on Hamiltonian Mechanics, Integrability and Chaotic Behavior,* edited by J. Seimenis (Plenum, New York, 1994), p. 261.

[4] M. N. Vrahatis, J. Comput. Phys. **119**, 105 (1995).

[5] C. L. Goudas, Bull. Soc. Math. Grèce **2**, 1 (1961).

[6] G. Shearing, Ph.D. thesis, University of Manchester, 1960 (unpublished).

[7] J. M. Ortega and W. C. Rheinboldt, *Iterative Solution of Nonlinear Equations in Several Variables* (Academic, New York, 1970).

[8] M. N. Vrahatis and K. I. Iordanidis, Numer. Math. **49**, 123 (1986).

[9] M. N. Vrahatis, G. Servizi, G. Turchetti, and T. C. Bountis, CERN Report No. SL/93-06 (AP), 1993 (unpublished).

[10] M. N. Vrahatis, T. C. Bountis, and M. Kollmann, Int. J. Bifurc. Chaos (to be published).

[11] A. Eiger, K. Sikorski, and F. Stenger, ACM Trans. Math. Software **10**, 367 (1984).

[12] M. N. Vrahatis, Bull. Soc. Math. Grèce **27**, 161 (1986).

[13] J. M. Greene, J. Comput. Phys. **98**, 194 (1992).

[14] M. N. Vrahatis, ACM Trans. Math. Software **14**, 312 (1988).

[15] M. N. Vrahatis, ACM Trans. Math. Software **14**, 330 (1988).

[16] J. Stoer and R. Bulirsch, *Introduction to Numerical Analysis* (Springer-Verlag, New York, 1980).

[17] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes, The Art of Scientific Computing,* 2nd ed. (Cambridge University Press, New York, 1992).